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Chaos in cosmology

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Abstract

We consider open cosmological models with multiply connected spacelike slices. In these universes there is a finite, convex region in which the world lines turn chaotic.

We point out some physical effects of the topology in wave fields, like particle creation-annihilation, violation of the space-reflection symmetry, and fluctuations in the temperature of the microwave background. The evolution of classical and quantum mechanical probability densities under the mixing horospherical flow is discussed.

Keywords: Cosmology; Chaos; Nonlinear Dynamics; Quantum Fields; Statistical Mechanics

1. INTRODUCTION

Cosmology - so far as the global structure of the universe is concerned - has always been somewhat at odds with the basic criterion of verifiability, that we rightly impose, since Galilei's time, onto a physical discipline.

This has lead to a very cautious attitude of many eminent physicists toward cosmological modeling. So, for example, the idea that the universe is infinite has been rejected on the grounds that we will not be able to look at infinity, and to verify what is happening there[1]. Other quite attractive arguments, like the atomicity of matter[2], or Mach's principle[3], were put forward to plead for the closure of space. A noticeable exception is Ref. 4. I think it is also fair to say, that closed universes are handier for heuristic reasoning[5,6].

However, there are also disadvantages. First of all, closed universes are very rigid, the only thing that is allowed to vary is the expansion factor, which defines the length unit at a given instant of time on the spacelike slices. The predictive records of such universes, tied to the principles of perfect homogeneity and isotropy, are extremely poor. It may also be appropriate to mention here that over the centuries most natural philosophers attached to the concept 'Universe' the attribute 'infinite', and some kind of non-trivial evolution toward a non-trivial end.

Although we are not able to look at infinity, there could nevertheless be the possibility to test infinite cosmological models, provided that local, microscopic phenomena are influenced by the global topological and metrical structure of the universe. Consequently, if we start with this assumption, we must abandon the reasoning 'we do not know the global structure, and therefore we assume the simplest possible'. Instead we have to try different topological and metrical scenarios, and to find traces of them in the quantum fields, and then we have to see what makes the difference.

It is largely agreed today that the rough overall evolution of the universe should be described by Riemannian geometry, but virtually nothing is known about the topological structure of space-time. Though allusions to this question emerge in the literature for a long time, neither philosophers nor scientists have been much inspired by it (compare the introduction of Ref. 7).

In Refs. 7-14 we suggested a practical way to gain more insight into the global topological structure of the universe. We assumed a given topological scenario, and tried to find physical effects of the topology. Do particles, rays, fields, and galaxies behave qualitatively differently in infinite universes of different connectivity? In the long run they do.

Our basic assumption is that the universe is open, infinite, without boundaries, and locally endowed with a Robertson-Walker line element of negative spatial curvature. The topology of the spacelike slices that we investigated was either $I \times D_N$, or $I \times S$, where I is a finite open interval, D_N is a disk with N smaller disks cut out, $N \geq 1$, and S is a Riemann surface of genus $g \geq 2$. The product $I \times D_N$ is topologically a thickened surface, $I \times D_N$ is a solid handlebody. These structures are the two major classes of topological three-manifolds that can be used as spacelike slices of universes with the above mentioned properties.

Though these spaces have a very compact topological representation, they are actually infinite, without physical boundaries, because the metric on them gets singular at the topological boundaries. A particle or ray starting from somewhere in the interior will not reach the boundary within a finite time, irrespectively of the choice of the expansion factor $a(\tau)$ in the metric[7,8,13].

Though contemporary cosmology originated from general relativity, the input that comes from this theory is rather small and qualitative: the existence of a space-time line-element, the geodesic principle, the principle of general covariance. Additional assumptions are needed to fix the actual form of the line-element: the principles of homogeneity and isotropy assure that the 3-space is of constant curvature at any given instant of time. That determines the line-element apart from a time-dependent expansion factor $a(\tau)$ that sets the length scale in the 3-space, and apart of the sign k ($k= -1, 0,$ or $+1$) of the curvature of the 3-space. The line-element can then be cast into the form $ds^2 = -c^2 d\tau^2 + 4a^2(\tau)\left(1+k|\mathbf{x}|^2/R^2\right)^{-2} dx^2$. That is just the Minkowski metric with Euclidean 3-space replaced by the $a(\tau)$-scaled, constantly curved 3-space. Such line-elements are usually referred to as of Robertson-Walker (RW) type[6].

What have Einstein's equations then to contribute to cosmology? Clearly they do not determine the metric, even not locally, because for that we would have to know the energy-momentum tensor of the universe, which is not the case. If we insert the RW-metric into the Einstein equations we can however express the energy and the pressure density of the universe as a function of $a(\tau)$ and its derivatives. The requirement of positivity of energy and pressure gives then restrictions on the asymptotic behavior of $a(\tau)$, for example in the case $k = -1$ positivity would require $a(\tau) \sim \Lambda\tau$ for $\tau \rightarrow \infty$. (See Ref. 4 for striking criticism on these predictions). In the limit $\tau \rightarrow 0$ (or $\tau \rightarrow -\infty$, if one believes in an infinite past) positivity arguments of this kind are not reliable, because in this limit either pressure or energy density diverge to infinity, and the classical Einstein equations are not supposed to give even qualitatively correct results in this regime. Anyhow, the expansion factor, the pressure and the energy density remain unknown functions, and even if one postulates ad hoc and in addition a universal thermodynamic equation of state this situation is not remedied. I do not want to deny here the considerable heuristic and phenomenological value of such supplements, however we are here more interested in the underlying structure of space-time, and on its bearing on the microscopic dynamics, and about that these equations have little to say.

Up to now we have only addressed the local, metric structure, the line element. In cosmology a much more important question is the global structure of space-time, the global topology, which is clearly left totally undecided by general relativity, being a purely metrical theory. Usually this question is settled in textbooks by postulating maximal symmetry, the existence of a continuous six-parameter symmetry group transitively acting on the space-like slices. This postulate restricts then the possible topologies to be either Euclidean space ($k=0$), or the 3-sphere ($k=1$), or projective 3-space ($k=1$), or a shell of the Minkowski hyperboloid ($k=-1$). Moreover there is only one metric of constant curvature that one can impose on these topologies. Maximal symmetry implies perfect homogeneity and isotropy.

This principle of maximal symmetry of the 3-space is of course very convenient, because it fixes once and for all both the topology and the geometry. However we regard it as too rigid, too static, unnecessary restrictive, and we suggest to relax it by requiring instead of maximal symmetry constant (negative) curvature. Moreover we require that the 3-space is infinite. As pointed out in the preceding papers this gives rise to a variety of possible topologies. Moreover the topology and the requirement of constant curvature does not any more determine the metric uniquely, which therefore may itself vary in time, apart from the scaling with the expansion factor. Locally, on sufficiently small coordinate patches we have still the six-parameter group of isometries, and the space is locally homogeneous and isotropic. For a more pictorial description of isotropy, homogeneity, and symmetry in this context we refer to Ref.12. Finally I mention that as soon as the topology is non-trivial there does not any more exist a continuous group of global isometries, in a sense it is replaced by the discrete covering group of the 3-space manifold.

Because the actual impact of Einstein's equations is rather small, one has to pose the question in how far Riemannian geometry enters at all in cosmologies with constantly curved spaces. Though it provides the underlying structure of the principle of general relativity, the actual use of Riemannian geometry is trivialized by the simplicity of the RW-line element. At first, the time-coordinate in the line-element is extremely distinguished, which highly encourages the use of 3-dimensional formalism, using time as a parameter. The second reason is that the constant sectional Gaussian curvature of the 3-space makes the use of the Riemann curvature tensor superfluous, and renders the 3-space more to an object of elementary geometry.

In the following Sections we report on some phenomena that arise in infinite and multiply connected cosmologies. For all the technical details the reader is referred to Refs. 8-10 and 14.

2. THE CENTRE OF THE UNIVERSE

Though the Universe is infinite and unbounded it has a centre[8,12]. There is a finite, three-dimensional convex domain in which the classical world lines are mixing. The diameter of this domain defines a natural length scale. This domain is the intersection of the convex hull of the limit set $\Lambda(\Gamma)$ of the covering group Γ (depicted in Figures 1 and 2) with the polyhedron

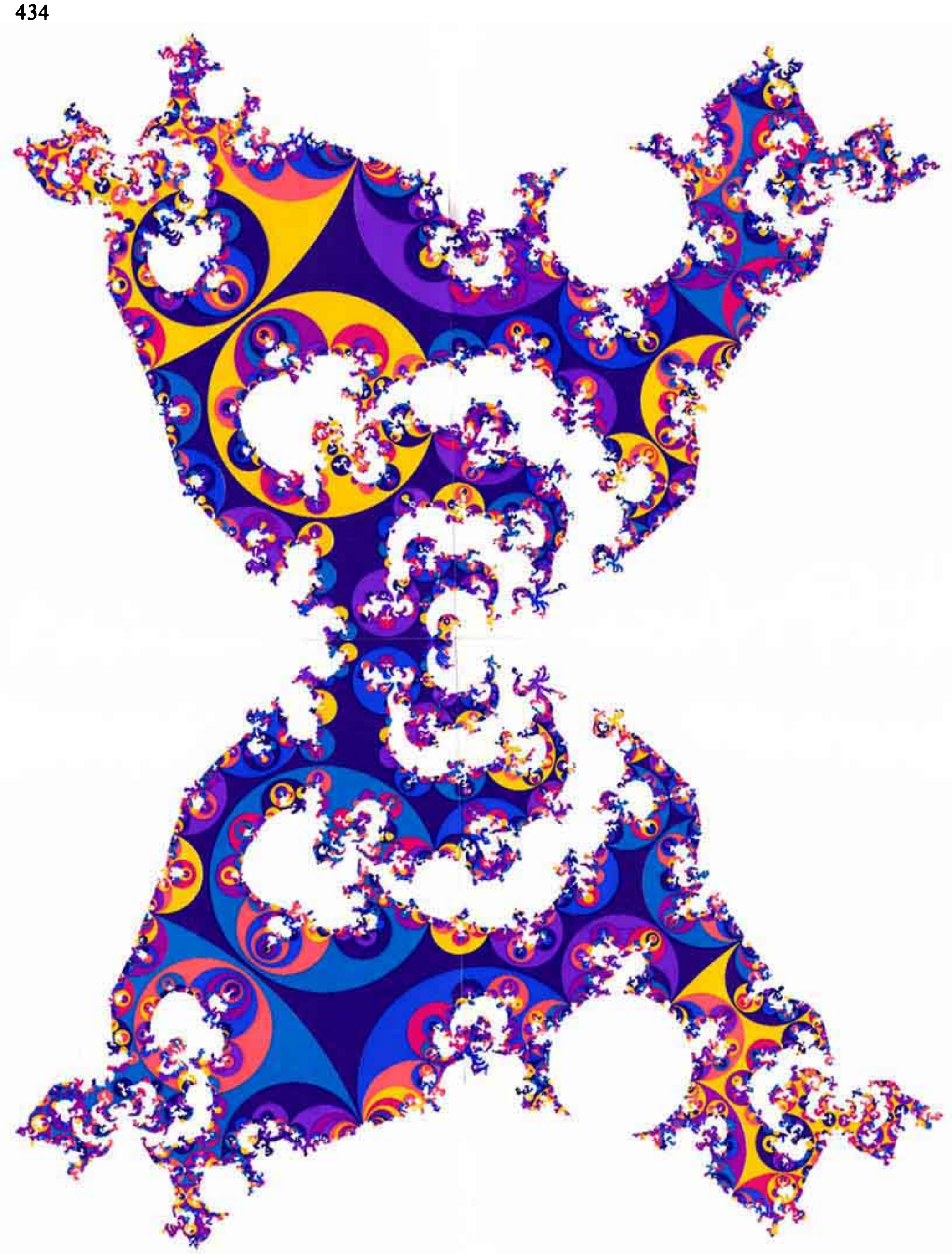

Figure 1. Tiling on the plane at infinity of the Poincaré half-space H^3. A spacelike slice (F, Γ) is realized in H^3 as a polyhedron F with a face-identification[15,16]. The identifying transformations generate a discrete group Γ. This group Γ applied to the polyhedron gives a tessellation $\Gamma(F)$ of H^3 with polyhedral images. This tessellation induces a tiling on the boundary of H^3. There is also a tiling exterior to the Jordan curve. Figures 1 and 2 stem from manifolds which are topologically equivalent but non-isometric ($I \times S$, $g(S) = 49$, $\delta(\Lambda) \approx 1.5$).

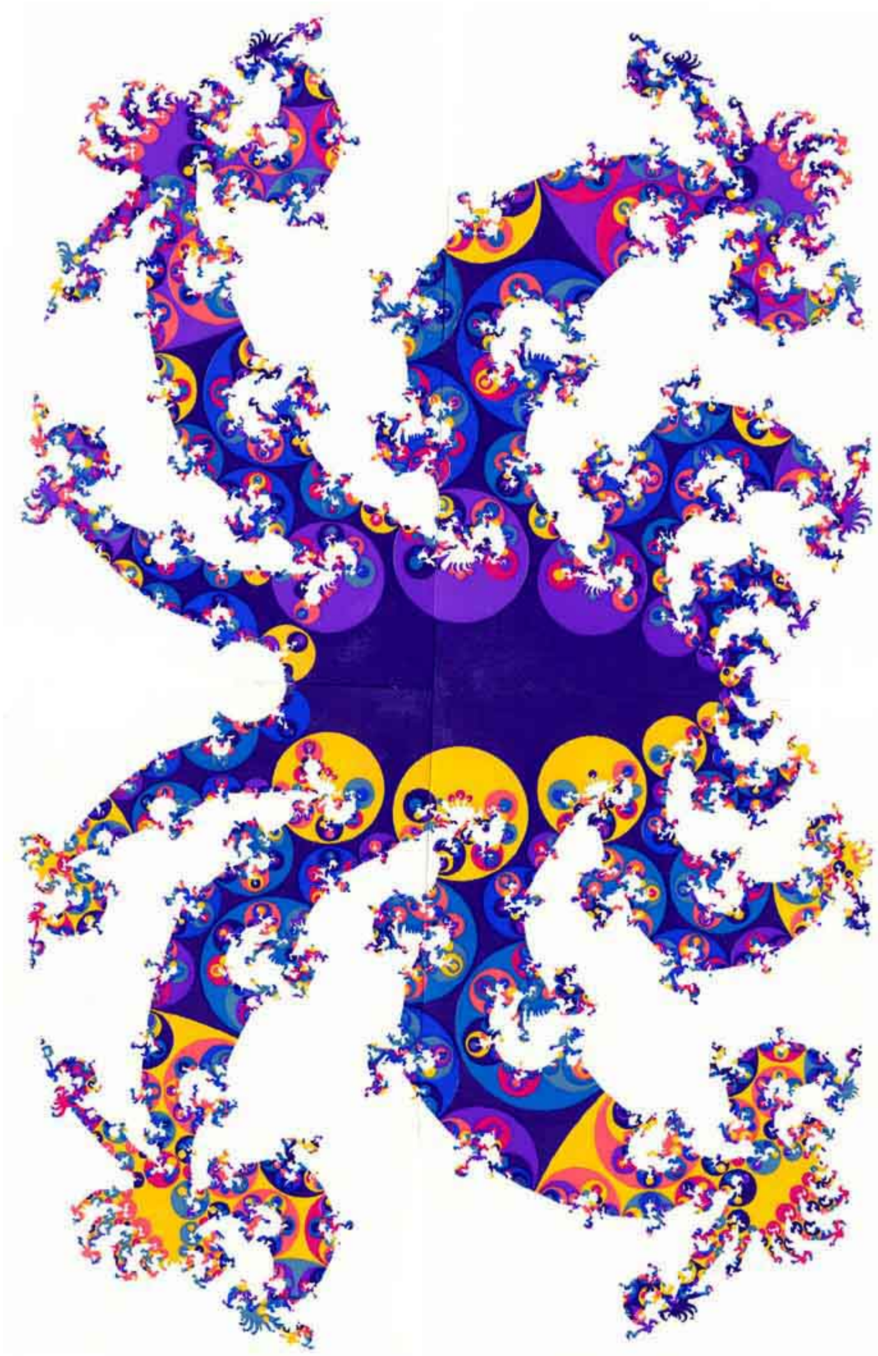

Figure 2. The singular curve $\Lambda(\Gamma)$ is the set of accumulation points of the polyhedral tiling $\Gamma(F)$ of H^3. The half-space H^3 is the universal covering space of a multiply connected three-manifold (F,Γ) that represents a spacelike slice of the universe at a given instant of time. The centre $C(\Lambda)\backslash\Gamma$ of the three-manifold is the intersection of the hyperbolic convex hull $C(\Lambda)$ of $\Lambda(\Gamma)$ with the polyhedron F. Its boundaries on the polyhedral faces are identified by elements of the covering group Γ. Trajectories that are mixing in the centre of the spacelike slices are unstable and have covering trajectories with an end point in $\Lambda(\Gamma)$.

that represents the three-space manifold in the covering space. More precisely, the centre is the quotient $C(\Lambda)\backslash\Gamma$. Chaotic trajectories have covering trajectories with at least one end-point in $\Lambda(\Gamma)$. For more elaboration on that see the figure captions and Refs. 8,15,16.

The centre is not a manifold, its surface is pleated, and the pleats may even accumulate. The surface however is not fractal, though it is generated by a fractal set. It consists of smooth pieces on totally geodesic planes that are joint along the pleats. In the Klein model of hyperbolic geometry, where hyperbolic convexity coincides with Euclidean convexity, it is not difficult to get an intuitive picture of the boundary of the convex hull, a kind of Euclidean polyhedron with infinitely many edges, which accumulate from time to time.

This domain of chaoticity, and the appearance of a new length scale[14] (given by its diameter) is absent in the contemporary standard models[6] of cosmology - it is a typical consequence of the multiple connectivity of the three-space. Clearly one is very tempted to make this mixing property of the classical world lines responsible for the more *or less* uniform distribution of the galaxies.

Though the limit set at infinity of H^3 may have a Hausdorff dimension δ close to two ($\delta(\Lambda)\approx 1.5$ in Figures 1 and 2), its Riemann-Lebesgue measure is zero. We think here the plane at infinity stereographically mapped onto the Riemann sphere, and take the spherical measure. Therefore the probability that an arbitrary trajectory in the three-space, if lifted into the covering space H^3, has one of its end points in the limit set is zero. Moreover, an arbitrarily small perturbation will move it out of the limit set. However, if the covering trajectory has its end point close to the limit set, it will loop a very long time in or close to the centre $C(\Lambda)\backslash\Gamma$ of the three-space, and accordingly a large proportion of the trajectories will be mixing in every practical sense, before they ultimately leave the centre and move toward infinity.

In the centre scalar wave fields can get localized (and trajectories can get trapped): The Klein-Gordon equation has, independent of the mass and the coupling to the curvature, square-integrable bound state solutions which are stable (e.g. against small metrical perturbations), in strong contrast to the classical counterpart. This bound state is well separated from the continuous spectrum. Outside the centre this bound state wave field decays exponentially. Interestingly, if one attaches spin to the particles they can easily escape, no bound states I could spot in the spectra of electromagnetic and Dirac fields[10,17].

3. CLASSICAL AND QUANTUM DISPERSION

One of the most remarkable features of simply connected Robertson-Walker cosmologies of negative spatial curvature is the instability of the classical geodesic trajectories, the probabilistic character of world lines. The most efficient and quantitative way to describe such systems that are highly sensitive with respect to the choice of the initial conditions is that of statistical mechanics.

We developed a probabilistic description of this instability, similar to the Liouville equation, but in a manifestly covariant, non-Hamiltonian form[9]. To achieve this we introduced the concept of a horospherical geodesic flow of expanding bundles of parallel world lines. We constructed an invariant measure and a covariant evolution equation for the probability density on which this flow acts.

The orthogonal surfaces to these bundles of trajectories are horospheres, closed surfaces in 3-space, touching the boundary at infinity of hyperbolic space, where the flow lines emerge. These horospheres are just the wave fronts of spherical waves which constitute a complete set of eigenfunctions of the Klein-Gordon equation. This fact suggests to compare the evolution of the quantum mechanical density with the classical one on which expanding bundles of geodesic flow lines act. We found asymptotic identity in the asymptotically flat region and in periods of adiabatic expansion, when no particle production processes occur. This lead us furthermore to study the time behavior of the dispersion of the energy and the coordinates and of the energy-time uncertainty relation, and we found again identity in the late stage of the cosmic evolution. This identity can persist in the early phase of the expansion with a rapidly varying scale factor, provided the fields are conformally coupled to the curvature.

From that we learn that the dispersion of the quantum mechanical wave packet is not due to some mysterious features of microscopic particles, but rather a consequence of the quantum mechanical description, which is clearly of a purely statistical nature - just as we describe the classical unstable one-particle system by a probability density.

4. SELF-INTERFERENCE: THE VIOLATION OF THE SPACE REFLECTION SYMMETRY

In a multiply connected space a dispersing probability density will start at some time to overlap with itself, for example it may wrap around a handle of the manifold. At this point the analogy between unstable classical mechanics and quantum mechanics breaks down, and new features, described by a new fundamental statistical constant, Planck's quantity h, emerge.

In the classical case the overlap of the probability density is purely additive. In the quantum description the overlap of the wave function with itself gives rise to self-interference. If we square the self-interfering wave packet, then the resulting density shows an interference pattern, which is clearly absent in the classical case.

The classical probability density describes a flow of non-interacting particles, which differ only by their initial conditions. The self-interference could be imagined as a statistical self-interaction in this classical flow, a force able to create and annihilate particles whenever distant parts of the density coalesce, generating so the interference pattern.

There is now substantial evidence that the charge conjugation C, the time reversal T and the space reflection P (parity) are not generic symmetries on the microscopic level. (There is a natural and straightforward way to construct space reflections on the multiply connected spacelike slices, just by combining a space inversion of hyperbolic space H^3 with the universal covering projection[17].) Quantum field theory can cope easily with that, but to describe it one has to add on purpose symmetry breaking interaction terms to the Lagrangians. Concerning neutrinos, the situation is even worse. If we describe them by four-component spinors we must exclude without a priori reason half of the possible solutions of the Dirac equation, and two-component neutrino equations do not fit nicely into the general scheme.

The T-symmetry is violated because of the expansion of the space. That is actually not a surprising thing, and occurs in practice also in classical probabilistic systems. In order to render such classical systems time-symmetric one would have to prepare initial and end value conditions with infinite precision. In fact, geodesic motion in this context is a good example for that[7].

The violation of the space-inversion symmetry has no classical counterpart. It is an interference phenomenon, that stems from the fact that in a multiply connected space a P-reflected wave packet can overlap with itself. Finally, C is still a good symmetry of the free Dirac equation, but the unitarity of CP and noticeably CPT is violated. Clearly one is tempted to speculate if this can lead, if combined with particle annihilation-creation processes (which occur likewise in the free Dirac equation[17]) to a dynamic generation of the baryon asymmetry in the universe[19]. For neutrinos CP is likewise a broken symmetry in a multiply connected universe. Finally, if one associates particles with microscopic topological excitations on the spacelike slices[3], one wonders whether the CP violation in kaon systems could not be a result of topological self-interference.

5. PARTICLE PRODUCTION AND ANNIHILATION PROCESSES DUE TO GLOBAL METRICAL DEFORMATIONS OF SPACE-TIME, AND FLUCTUATIONS IN THE TEMPERATURE OF THE MICROWAVE BACKGROUND

If the three-space is infinite and multiply connected it can undergo global metrical deformations, in a way that the Gaussian curvature stays unaltered, say $-1/R^2$, where R is the curvature radius. I emphasize that this is not the case in a simply connected or finite, closed universe. One may imagine such a deformation by means of a surface embedded in three-dimensional Euclidean space, that is bent without being torn. The analogy is however not perfect, because the Euclidean metric induced on such a surface is usually not of constant curvature, and the curvature radii vary during the deformation. Manifolds, (surfaces or higher dimensional) of constant curvature are modeled as polygons or polyhedra in Euclidean or hyperbolic or spherical (covering) space (of the same dimension). One has then to imagine the polyhedral faces glued together in pairs, by transformations of the symmetry group of the covering space.

The metric on the polyhedron is that of the (hyperbolic) covering space, and so the constant (negative) curvature is inherited on the manifold. Global metrical deformations are then simply deformations of the polyhedron. Another equivalent way to generate such deformations is to keep the polyhedron fixed and to add perturbations (namely symmetric traceless tensor fields that are invariant with respect to the covering group) to the hyperbolic metric[10,20].

During periods of such global metrical deformations of the spacelike slices particle annihilation-creation processes occur in relativistic quantum fields, for example in spinor fields satisfying the free Dirac equation. (In the above covering space construction with the spacelike slices represented as polyhedra, we have to impose on the wave fields periodic boundary conditions with respect to the identified faces.) A wave packet composed of positive frequencies will receive admixtures of negative frequencies during such a deformation. This happens also in conformally coupled fields, like neutrinos.

In the electromagnetic field such deformations, if they are sufficiently adiabatic, distort a little the dependence of the frequencies of the horospherical elementary waves from the spectral variables. This distortion manifests then in a small angular and time dependence of the temperature in the Planck distribution of the microwave background[10,20].

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